The elements in crystal bases corresponding to exceptional modules*

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Abstract

According to the Ringel-Green Theorem([G],[R1]), the generic composition algebra of the Hall algebra provides a realization of the positive part of the quantum group. Furthermore, its Drinfeld double can be identified with the whole quantum group([X],[XY]), in which the BGP-reflection functors coincide with Lusztig's symmetries. We first assert the elements corresponding to exceptional modules lie in the integral generic composition algebra, hence in the integral form of the quantum group. Then we prove that these elements lie in the crystal basis up to a sign. Eventually we show that the sign can be removed by the geometric method. Our results hold for any type of Cartan datum.

1 Introduction

Let Δ be a symmetrizable generalized Cartan matrix, or $\Delta = (I, (-,-))$ a Cartan datum in the sense of Lusztig [L1], $\mathfrak{g}$ the corresponding symmetrizable Kac-Moody algebra. We have the Drinfeld-Jimbo quantized enveloping algebra, or the quantum group, $U = U_q(\mathfrak{g})$ attached to the Cartan datum Δ. Lusztig gave it a series of important automorphisms, now called Lusztig's symmetries. By applying Lusztig's symmetries and the induced action of the braid group on U^+, Lusztig found an algebraic approach to construct the canonical basis of U^+ in finite type. If Δ is of infinite type, there are much more root vectors beyond the set obtained by applying the braid group action on Chevalley generators of U^+. However Lusztig's geometric method by using perverse sheaves and intersection cohomology to construct the canonical bases works for general infinite type. A different algebraic approach due to Kashiwara works for arbitrary type. He constructed the crystal basis and the global crystal basis (= canonical basis) of the negative part U^- of the quantum group. Roughly speaking, the crystal basis is a good basis of the quantum group at $q = 0$.

Given a Cartan datum Δ, there is a finite dimensional hereditary k-algebra Λ to realize it, where k is a finite field. Then we have the corresponding Hall algebra $\mathscr{H}(\Lambda)$. According to the Ringel-Green Theorem(see [G],[R1]), the generic composition algebra $\mathscr{C}(\Delta)$ of $\mathscr{H}(\Lambda)$, precisely of the Cartan datum Δ, provides a realization of the positive part U^+ of the quantum group corresponding to Δ.

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With the comultiplication of $\mathscr{H}(\Lambda)$ given by Green in [G], it is natural to obtain a Hopf algebra structure of $\mathscr{H}(\Lambda)$ by adding a torus, and then to consider the Drinfeld double of the Hall algebra (see [X]). Therefore, the Drinfeld double of the generic composition algebra $\mathcal{D}_{\mathscr{C}}(\Delta)$ provides a realization of the whole U. This realization builds up a bridge between the quantum groups and the representation theory of hereditary algebras. Especially, it connects Lusztig's symmetries with BGP-reflection functors (see [XY]), so this two important operators can be considered simultaneously.

In this article, we consider a special family of elements $\{\langle u_\lambda\rangle|\lambda \in \mathcal{E}\}$ in the Hall algebra, where $\mathcal{E}$ is the set of isomorphism classes of all exceptioal Λ-modules. These elements are much more than the elements obtained by applying the braid group action on Chevalley generators of U^+. From the work of [Z] and [CX] we know that these elements lie in the generic composition algebra $\mathcal{D}_{\mathscr{C}}(\Delta)$. Our first result asserts in Theorem 6.1 that each $\langle u_\lambda\rangle$ lies in the integral generic composition algebra $\mathscr{C}_{\mathbb{Z}}(\Delta)$, hence in the integral form of the positive part of the quantum group $U^+_{\mathbb{Z}}$ (by identifying $\mathcal{D}_{\mathscr{C}}(\Delta)$ with U).

The main goal of us is to relate the exceptional modules with Kashiwara's crystal bases. For convenience, we use the the crystal structure $(L(\infty), B(\infty))$ in U^+ instead of in U^-. Our main result is (see Theorem 6.2) that the image of $\langle u_\lambda\rangle$ in $L(\infty)/qL(\infty)$ lies actually in the crystal basis $B(\infty)$ up to a sign. In the last section we remove the sign by comparing with Lusztig's geometric method. Therefore the image of $\langle u_\lambda\rangle$ in $L(\infty)/qL(\infty)$ always belongs to $B(\infty)$.

The organization of this paper is as follows: In Section 2, we review some basic facts of quantum groups and crystal bases. Then the polarization, which will be called Kashiwara's pairing, is defined in the positive part of quantum groups. For the details, see [L1] and [K]. In Section 3, we first give the definitions of Hall algebras and composition algebras. Following [X], we concisely restate the Drinfeld double structure of Hall algebras. Also, we get some important operators r'_i in the Hall algebras. Then r'_i is the same as the derivation operators f'_i when we identify the generic composition algebra with U^+. The aim of Section 4 is to obtain $\langle u_\lambda\rangle$ corresponding to preprojective or preinjective modules from the simple modules, after establishing the isomorphism between Lusztig's symmetries and reflection functors. The results come from [BGP], [R5] and [XY]. Section 5 gives a review of an algorithm in [CX]. This algorithm comes from a result of Crawley-Boevey [CB], to state that any exceptional module can be obtained inductively from simple modules using braid group actions on exceptional sequences. The main results will be stated in Section 6. In Section 7 we prove the first main result by a combination of algorithms in Section 4 and 5. Then in Section 8, we introduce Ringel's pairing in U^+, and compare it with Kashiwara's pairing. Our second main result follows from direct calculations of Ringel's pairing. However, we need to remove the sign, which will be done using geometric methods in the last section.

2 Quantum groups and crystal bases

2.1 Quantum groups. Let $(I, (-,-))$ be a *Cartan datum* in the sense of Lusztig. i.e. I is a finite set and $(-,-)$ is a symmetric bilinear form $\mathbb{Z}[I]\times\mathbb{Z}[I] \to \mathbb{Z}$ which satisfies the following conditions:

(a) $(i,i) \in \{2, 4, 6, ...\}$ for any $i \in I$.

(b) $2(i,j)/(i,i) \in \{0,-1,-2,...\}$ for any $i \neq j$ in I.

Let $Q = \mathbb{Z}[I]$, $Q_+ = \mathbb{N}[I]$. Q is called the *root lattice*. For any $i \in I$ define $s_i : Q \to Q$ by $s_i(\mu) = \mu - \dfrac{2(\mu,i)}{(i,i)}i$. s_i is called a *simple reflection*. The *Weyl group* W is defined to be the group generated by all the simple reflections.

Note that we can identify a Cartan datum $(I,(-,-))$ with a *symmetrizable generalized Cartan matrix* $\Delta = (a_{ij})_{i,j}$ by setting $a_{ij} = 2(i,j)/(i,i)$. Let $\varepsilon_i = (i,i)/2$, then $(\varepsilon_i)_i$ is the minimal symmetrization. We have the corresponding Kac-Moody algebra $\mathfrak{g}$.

Let $\mathbb{Q}(v)$ be the function field in one variable v over $\mathbb{Q}$. The Drinfeld-Jimbo quantum group $U = U_q(\mathfrak{g})$ is defined to be the associative $\mathbb{Q}(v)$-algebra with generators E_i, F_i, K_μ, ($i \in I$ and $\mu \in Q$) subject to the relations:

$$K_\nu K_\mu = K_\mu K_\nu = K_{\mu+\nu}, \quad K_0 = 1,$$

$$K_\mu E_j = v^{(\mu,j)} E_j K_\mu, \quad K_\mu F_j = v^{-(\mu,j)} F_j K_\mu$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{v_i - v_i^{-1}},$$

$$\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{\varepsilon_i} E_i^t E_j E_i^{1-a_{ij}-t} = 0, (i \neq j)$$

$$\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{\varepsilon_i} F_i^t F_j F_i^{1-a_{ij}-t} = 0, (i \neq j).$$

where $v_i = v^{\varepsilon_i}$ and we use the notations

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}} = v^{n-1} + v^{n-3} + \cdots + v^{-n+1},$$

$$[n]! = \prod_{r=1}^{n} [r], \quad \begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}.$$

and for any polynomial $f \in \mathbb{Z}[v, v^{-1}]$ and an integer a, we denote by f_a the polynomial obtained from f by replacing v by v^a.

The following elementary properties of U are well known:

(a) U has a triangular decomposition

$$U \cong U^- \otimes U^0 \otimes U^+,$$

where U^+ (resp. U^-) is the subalgebra of U generated by E_i (resp. F_i), $i \in I$, and U^0 is the subalgebra generated by $K_i^{\pm}$,$i \in I$.

(b) U^+ and U^- are Q_+-graded algebras, i.e.

$$U^+ = \bigoplus_{\nu \in Q_+} U_\nu^+, \quad U^- = \bigoplus_{\nu \in Q_+} U_{-\nu}^-.$$

where $U_\nu^{\pm} = \{u \in U^{\pm} | K_i u K_i^{-1} = v^{\pm(i,\nu)} u, \text{for any } \ i \in I\}$.

(c) U has a Hopf algebra structure (See [L1]).

Lusztig introduced an important family of automorphisms $T''_{i,1} : U \to U$ called the *symmetries*. They are defined by

$$T''_{i,1}(E_i) = -F_i K_i^{\varepsilon_i},$$

$$T''_{i,1}(F_i) = -K_i^{-\varepsilon_i} E_i,$$

$$T''_{i,1}(E_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{-r\varepsilon_i} E_i^{(s)} E_j E_i^{(r)}, \text{ for } i \neq j$$

$$T''_{i,1}(F_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{r\varepsilon_i} F_i^{(s)} F_j F_i^{(r)}, \text{ for } i \neq j$$

$$T''_{i,1}(K_\mu) = K_{s_i\mu}.$$

The inverse of $T''_{i,1}$ is $T'_{i,-1}$ (See [L1]).

2.2 Crystal bases of U^+. We will briefly recall the definition of crystal bases following Kashiwara [K]. However, for later convenience, we will state the results in U^+ rather than U^-.

There are two operators f'_i and f''_i on U^+. They can be defined inductively as following:

$$f'_i(1) = f''_i(1) = 0$$

$$f'_i(E_j) = \delta_{ij}, \quad f'_i(E_j P) = v_i^{a_{ij}} E_j f'_i(P) + \delta_{ij} P,$$

$$f''_i(E_j) = \delta_{ij}, \quad f''_i(E_j P) = v_i^{-a_{ij}} E_j f''_i(P) + \delta_{ij} P.$$

According to [K] we have $U^+ = \bigoplus_{n\geq 0} E_i^{(n)} \ker f'_i$. Hence we can define the $\mathbb{Q}(v)$-linear maps $\tilde{E}_i$ and $\tilde{F}_i$ of U^+ by

$$\tilde{E}_i(E_i^{(n)} u) = E_i^{(n+1)} u,$$

$$\tilde{F}_i(E_i^{(n)} u) = E_i^{(n-1)} u.$$

for any $u \in \ker f'_i$.

$\tilde{F}_i$ and $\tilde{E}_i$ are called *Kashiwara's operators.*

Let $A = \mathbb{Q}[[v^{-1}]] \cap \mathbb{Q}(v)$. A pair (L, B) is called a *crystal basis* of U^+ if it satisfies the following conditions:

(1) L is a free A-submodule of U^+ such that $U^+ \cong \mathbb{Q}(v) \otimes_A L$.

(2) B is a basis of the $\mathbb{Q}$-vector space $L/v^{-1}L$.

(3) Let $L_\nu = L \cap U_\nu^+$ and $B_\nu = B \cap (L_\nu/v^{-1}L_\nu)$, we have $L = \bigoplus_{\nu\in Q_+} L_\nu$, $B = \bigsqcup_{\nu\in Q_+} B_\nu$.

(4) $\tilde{E}_i L \subset L$ and $\tilde{F}_i L \subset L$ for any i. (Therefore $\tilde{E}_i$ and $\tilde{F}_i$ act on $L/v^{-1}L$)

(5) $\tilde{F}_i B \subset B \cup \{0\}$ and $\tilde{E}_i B \subset B$.

(6) For any $b \in B$ such that $\tilde{F}_i b \in B$, we have $\tilde{E}_i \tilde{F}_i b = b$.

The following theorem asserts the existence of the crystal basis in U^+.

Theorem 2.1. *Let $L(\infty)$ be the A-submodule of U^+ generated by $\tilde{E}_{i_1}\tilde{E}_{i_2}\cdots\tilde{E}_{i_l}\cdot 1$ and $B(\infty)$ be the subset of $L(\infty)/v^{-1}L(\infty)$ consisting of the nonzero vectors of the form $\tilde{E}_{i_1}\tilde{E}_{i_2}\cdots\tilde{E}_{i_l}\cdot\bar{1}$.*

Then $(L(\infty), B(\infty))$ is the crystal basis of U^+.

2.3 A characterization of $B(\infty)$. Kashiwara has given a nice characterization of the crystal basis using the $\mathbb{Z}$-form and a paring $(-,-)_K$.

Proposition 2.2. *(a) There is a unique non-degenerate symmetric $\mathbb{Q}(v)$-bilinear pairing $(-,-)_K$ on U^+ such that*

$$(1,1)_K = 1,$$

$$(E_i x, y)_K = (x, f_i'(y))_K.$$

(b) We have $(L(\infty), L(\infty))_K \subset A$.

Part (b) of this proposition implies that the form $(-,-)_K$ induces a $\mathbb{Q}$-bilinear form $(-,-)_{K,0}$ on $L(\infty)/v^{-1}L(\infty)$:

$$(x + v^{-1}L(\infty), y + v^{-1}L(\infty))_{K,0} = (x,y)_K + v^{-1}A$$

for any $x, y \in L(\infty)$.

Proposition 2.3. *(a) For any $b_1, b_2 \in B(\infty)$, $(b_1, b_2) = \delta_{b_1 b_2}$, i.e. $B(\infty)$ is an orthogonal normal basis of $L(\infty)/v^{-1}L(\infty)$ with respect to the form $(-,-)_{K,0}$. In particular, $(-,-)_{K,0}$ is positive definite.*

(b) $L(\infty) = \{u \in U^+ | (u,u)_K \in A\}$.

Set $E_i^{(n)} = E_i^n/[n]_{\epsilon_i}!$, $F_i^{(n)} = F_i^n/[n]_{\epsilon_i}!$. Denote by $U_{\mathbb{Z}}$ the $\mathbb{Z}[v, v^{-1}]$-subalgebra of U generated by $F_i^{(n)}$, $E_i^{(n)}$ and K_μ, $(i \in I,\ \mu \in Q)$. Let $U_{\mathbb{Z}}^+$ (resp. $U_{\mathbb{Z}}^-$) be the $\mathbb{Z}[v, v^{-1}]$-subalgebra of U^+ (resp. U^-) generated by $E_i^{(n)}$ (resp. $F_i^{(n)}$). Then it is easy to see that

$$U_{\mathbb{Z}}^- = U_{\mathbb{Z}} \cap U^-, \quad U_{\mathbb{Z}}^+ = U_{\mathbb{Z}} \cap U^+.$$

Lusztig's symmetries also act nicely on $U_{\mathbb{Z}}$, so actually $T''_{i,1}$ and $T'_{i,-1}$ are automorphisms on $U_{\mathbb{Z}}$. (See [L1], 37.1.3)

We have $U_{\mathbb{Z}}^+$ is stable under f_i' and Kashiwara's operators $\tilde{E}_i$, $\tilde{F}_i$.

Set $L_{\mathbb{Z}}(\infty) = L(\infty) \cap U_{\mathbb{Z}}^+$, then $L_{\mathbb{Z}}(\infty)$ is stable under $\tilde{E}_i$ and $\tilde{F}_i$.

Proposition 2.4. *(a) $(-,-)_{K,0}$ is $\mathbb{Z}$-valued on $L_{\mathbb{Z}}(\infty)/v^{-1}L_{\mathbb{Z}}(\infty)$.*

(b) $L_{\mathbb{Z}}(\infty)/v^{-1}L_{\mathbb{Z}}(\infty)$ is a free $\mathbb{Z}$-module with $B(\infty)$ as a basis.

(c) $B(\infty) \cup (-B(\infty)) = \{u \in L_{\mathbb{Z}}(\infty)/v^{-1}L_{\mathbb{Z}}(\infty) | (u,u)_{K,0} = 1\}$.

3 Hall algebras and Drinfeld double

3.1 The Hall algebra of a hereditary algebra. Let Λ be a finite-dimensional hereditary k-algebra where k is a finite field of q elements. Denote the set of isomorphism classes of finite-dimensional Λ-modules by $\mathcal{P}$. We can choose a representative $V_\alpha \in \alpha$ for each $\alpha \in \mathcal{P}$. Given any Λ-modules M and N, we have the Euler form:

$$\langle M, N \rangle = \dim_k \mathrm{Hom}_\Lambda(M,N) - \dim_k \mathrm{Ext}_\Lambda(M,N).$$

$\langle M, N \rangle$ depends only on the dimension vectors $\underline{\dim} M$ and $\underline{\dim} N$ since Λ is hereditary, so we write $\langle \alpha, \beta \rangle = \langle V_\alpha, V_\beta \rangle$. The Euler symmetric form $(-,-)$ is

given by $(\alpha,\beta)=\langle\alpha,\beta\rangle+\langle\beta,\alpha\rangle$. So the Euler form and the Euler symmetric form are both defined on $\mathbb{Z}[I]$ where I is the set of isomorphism classes of simple Λ-modules. Then $\Delta=(I,(-,-))$ is a Cartan datum and any Cartan datum can be realized by the Euler symmetric form of a finite-dimensional hereditary k-algebra (See [R4]).

For $\alpha,\beta,\lambda\in\mathcal{P}$, let $g^{\lambda}_{\alpha\beta}$ be the number of submodules B of V_λ such that $B\cong V_\beta$ and $V_\lambda/B\cong V_\alpha$.

Let $v=\sqrt{q}$, the *Hall algebra* of Λ is a free $\mathbb{Q}(v)$-module whose basis consists of isomorphism classes of Λ-modules with multiplication defined as

$$u_\alpha u_\beta=v^{\langle\alpha,\beta\rangle}\sum_{\lambda\in\mathcal{P}}g^{\lambda}_{\alpha\beta}u_\lambda,\ \text{for all }\alpha,\beta\in\mathcal{P}.$$

We know $\mathscr{H}(\Lambda)$ is an associative $\mathbb{N}[I]$-graded $\mathbb{Q}(v)$-algebra with the identity element u_0 and the grading $\mathscr{H}(\Lambda)=\bigoplus_{r\in\mathbb{N}[I]}\mathscr{H}_r$ where $\mathscr{H}_r$ is the $\mathbb{Q}(v)$-span of the set $\{u_\lambda|\lambda\in\mathcal{P},\underline{\dim}V_\lambda=r\}$. The $\mathbb{Q}(v)$-subalgebra $\mathscr{C}(\Lambda)$ generated by $u_i,i\in I$ is called the *composition algebra* of Λ.

In this paper we are dealing with exceptional modules. A Λ-module V_λ ($\lambda\in\mathcal{P}$) is called *exceptional* if $\mathrm{Ext}_\Lambda(V_\lambda,V_\lambda)=0$, i.e. V_λ has no self-extension. For any exceptional module V_λ, we set $u^{(t)}_\lambda=(1/[t]!_{\varepsilon(\lambda)})u^t_\lambda$ in the Hall algebra, where $\varepsilon(\lambda)=\dim_k\mathrm{End}_\Lambda V_\lambda$. We have the following identities: $u^{(t)}_\lambda=(v^{\varepsilon(\lambda)})^{t(t-1)}u_{t\lambda}$, where $u_{t\lambda}$ corresponds to the direct sum of t copies of V_λ.

Now fix a Cartan datum Δ. Let $\overline{k}$ be the algebraic closure of k and for any $n\in\mathbb{N}$, $F(n)$ be a subfield of $\overline{k}$ such that $[F(n):k]=n$ .Then we have a finite-dimensional hereditary $F(n)$-algebra $\Lambda(n)$ corresponding to Δ. Thus we have a series of Hall algebras $\mathscr{H}=\mathscr{H}(\Lambda(n))$. Define a new ring $\Pi=\prod_{n>0}\mathscr{H}_n$, then $v=(v_n)_n\in\Pi$ where $v_n=\sqrt{|F(n)|}=\sqrt{q^n}$. Obviously v is in the center of Π and transcendental over $\mathbb{Q}$. Denote $u_i=(u_i(n))_n\in\Pi$ where $u_i(n)$ is the element of $\mathscr{H}(\Lambda(n))$ corresponding to the simple $\Lambda(n)$-module which lies in the class $i\in I$. The subring of Π generated by the elements v, v^{-1} and $u_i(i\in I)$, hence a $\mathbb{Q}(v)$-algebra, is called the *generic composition algebra* of the Cartan datum Δ. We will denote it by $\mathscr{C}(\Delta)$.

On the other hand, we have U^+, the positive part of the Drinfeld-Jimbo quantum group corresponding to the Cartan datum Δ. By Green [G] and Ringel [R1] we know that $\mathscr{C}(\Delta)$ is isomorphic to U^+ as associative $\mathbb{Q}(v)$-algebras, where u_i is sent to E_i for each $i\in I$. Therefore, corresponding to the $\mathbb{Z}$-form of quantum groups, we can define $\mathscr{C}_{\mathbb{Z}}(\Delta)$ to be the $\mathbb{Z}[v,v^{-1}]$-subalgebra of $\mathscr{C}(\Delta)$ generated by $u^{(t)}_i,i\in I,t\in\mathbb{N}$, which will be called the *integral generic composition algebra* of the Cartan datum Δ. Obviously $\mathscr{C}_{\mathbb{Z}}(\Delta)$ is isomorphic to $U^+_{\mathbb{Z}}$.

3.2 The Drinfeld double. In the Hall algebra $\mathscr{H}(\Lambda)$, we write $\langle u_\alpha\rangle=v^{-\dim_k V_\alpha+\varepsilon(\alpha)}u_\alpha$ for each $\alpha\in\mathcal{P}$. Here $\varepsilon(\alpha)=\langle\alpha,\alpha\rangle$, which is equal to $\dim_k\mathrm{End}_\Lambda V_\alpha$ when V_α is exceptional. Then $\mathscr{H}(\Lambda)$ can be viewed as a free $\mathbb{Q}(v)$-algebra with basis $\langle u_\alpha\rangle,\alpha\in\mathcal{P}$. The multiplication formula can be replaced by

$$\langle u_\alpha\rangle\langle u_\beta\rangle=v^{-\langle\beta,\alpha\rangle}\sum_{\lambda\in\mathcal{P}}g^{\lambda}_{\alpha\beta}\langle u_\lambda\rangle\quad\text{for all }\alpha,\beta\in\mathcal{P}$$

Now we introduce the extended Hall algebra $\mathcal{H}(\Lambda)$ by adding a torus to

$\mathscr{H}(\Lambda)$. Let $\mathcal{H}(\Lambda)$ be the free $\mathbb{Q}(v)$-module with the basis

$$\{K_\alpha\langle u_\lambda\rangle | \alpha\in\mathbb{Z}[I], \lambda\in\mathcal{P}\}.$$

and extend the multiplication by

$$K_\alpha\langle u_\beta\rangle = v^{(\alpha,\beta)}\langle u_\beta\rangle K_\alpha \quad \text{for all } \alpha\in\mathbb{Z}[I], \beta\in\mathcal{P}$$
$$K_\alpha K_\beta = K_{\alpha+\beta} \quad \text{for all } \alpha,\beta\in\mathbb{Z}[I]$$

Moreover, $\mathcal{H}(\Lambda)$ has been equipped with a Hopf algebra structure by Green's comultiplication and an antipode (See [G], [X]).

Let $\mathcal{H}^+(\Lambda)$ be the Hopf algebra $\mathcal{H}(\Lambda)$ above but we write $\langle u_\lambda^+\rangle$ for $\langle u_\lambda\rangle$ for all $\lambda\in\mathcal{P}$. Dually, we can define $\mathcal{H}^-(\Lambda)$ to be the free $\mathbb{Q}(v)$-module with the basis $\{K_\alpha\langle u_\lambda^-\rangle | \alpha\in\mathbb{Z}[I], \lambda\in\mathcal{P}\}$. $\mathcal{H}^-(\Lambda)$ has a similar Hopf algebra structure (See [X] or [XY]).

In view of [X], we obtain the *the reduced Drinfeld double* $\mathcal{D}(\Lambda)$ coming from a Hopf algebra structure of $\mathcal{H}^+(\Lambda)\otimes\mathcal{H}^-(\Lambda)$, by means of a skew Hopf paring on $\mathcal{H}^+(\Lambda)\times\mathcal{H}^-(\Lambda)$. Then there exists the reduced Drinfeld double $\mathcal{D}_{\mathscr{C}}(\Delta)$ of the generic composition algebra which is generated by $u_i^\pm, i\in I$, and $K_\alpha, \alpha\in\mathbb{Z}[I]$. Then $\mathcal{D}_{\mathscr{C}}(\Delta)$ has the triangular decomposition $\mathcal{D}_{\mathscr{C}}(\Delta) = \mathscr{C}^-(\Delta)\otimes T\otimes\mathscr{C}^+(\Delta)$, where $\mathscr{C}^-(\Delta)$ is the subalgebra generated by $u_i^-, i\in I$, $\mathscr{C}^+(\Delta)$ the subalgebra generated by $u_i^+, i\in I$, and T the torus algebra.

Theorem 3.1. *(See [X]) The map $\theta:\mathcal{D}_{\mathscr{C}}(\Delta)\to U$ by sending*

$$\langle u_i^+\rangle\to E_i, \langle u_i^-\rangle\to -v^{\varepsilon(i)}F_i, K_i\to K_i^{\varepsilon(i)}$$

for all $i\in I$ induces an isomorphism as Hopf $\mathbb{Q}(v)$-algebras.

3.3 Some Derivations. For $\alpha\in\mathcal{P}$, we define the following operators on $\mathscr{H}(\Lambda)$:

$$r_\alpha(\langle u_\lambda\rangle) = \sum_{\beta\in\mathcal{P}} v^{\langle\beta,\alpha\rangle+(\alpha,\beta)} g_{\beta\alpha}^\lambda \frac{a_\beta a_\alpha}{a_\lambda}\langle u_\beta\rangle$$

$$r'_\alpha(\langle u_\lambda\rangle) = \sum_{\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle+(\alpha,\beta)} g_{\alpha\beta}^\lambda \frac{a_\beta a_\alpha}{a_\lambda}\langle u_\beta\rangle$$

for all $\lambda\in\mathcal{P}$, where $a_\alpha = \mathrm{Aut}_\Lambda(V_\alpha)$.

The following lemma can be proved by direct calculation (similar to [CX], Prop 3.2).

Lemma 3.2. *For any $i\in I$ and $\lambda_1,\lambda_2\in\mathcal{P}$, we have*

$$r_i(\langle u_{\lambda_1}\rangle\langle u_{\lambda_2}\rangle) = \langle u_{\lambda_1}\rangle r_i(\langle u_{\lambda_2}\rangle) + v^{(i,\lambda_2)} r_i(\langle u_{\lambda_1}\rangle)\langle u_{\lambda_2}\rangle.$$

$$r'_i(\langle u_{\lambda_1}\rangle\langle u_{\lambda_2}\rangle) = v^{(i,\lambda_1)}\langle u_{\lambda_1}\rangle r'_i(\langle u_{\lambda_2}\rangle) + r'_i(\langle u_{\lambda_1}\rangle)\langle u_{\lambda_2}\rangle.$$

By the lemma above it is easy to see that

$$r'_i(1) = r'_i(\langle u_0\rangle) = 0,\ r'_i(\langle u_j\rangle) = \delta_{ij},$$

$$r'_i(\langle u_j\rangle\langle u_\lambda\rangle) = v^{(i,j)}\langle u_j\rangle r'_i(\langle u_\lambda\rangle) + \delta_{ij}\langle u_\lambda\rangle.$$

So if we restrict r'_i to the composition algebra $\mathscr{C}(\Lambda)$, we will have

$$r'_i(\langle u_j\rangle P) = v^{(i,j)}\langle u_j\rangle r'_i(P) + \delta_{ij}P, \quad \textit{for any } P\in\mathscr{C}(\Lambda)$$

Lemma 3.3. *When we identify U^+ with the generic composition algebra, we have $f_i' = r_i'$.*

Proof. Just compare the above formulas with the definition of f_i' in Section 2.2. $\square$

Also, there are two other derivations on $\mathscr{H}(\Lambda)$ which have nice properties. Namely, define

$$_{\alpha}\delta = \frac{(v^2)^{-\dim_k V_\alpha + \varepsilon(\alpha)}}{a_\alpha} r_\alpha', \;\; \delta_\alpha = \frac{(v^2)^{-\dim_k V_\alpha + \varepsilon(\alpha)}}{a_\alpha} r_\alpha.$$

The key properties of $_{\alpha}\delta$ and δ_α are as following (but note that here our definitions are slightly different from the original ones):

Proposition 3.4. *We consider the following linear maps:*

$$\begin{array}{rcl} \phi_1 : \mathscr{H}(\Lambda) & \longrightarrow & Hom_{\mathbb{Q}(v)}(\mathscr{H}(\Lambda), \mathscr{H}(\Lambda)) \\ \langle u_\lambda \rangle & \longrightarrow & _{\lambda}\delta \\ \phi_2 : \mathscr{H}(\Lambda) & \longrightarrow & Hom_{\mathbb{Q}(v)}(\mathscr{H}(\Lambda), \mathscr{H}(\Lambda)) \\ \langle u_\lambda \rangle & \longrightarrow & \delta_\lambda \end{array}$$

(1) ϕ_1 is an anti-homomorphism, i.e $\phi_1(\langle u_{\lambda_1}\rangle\langle u_{\lambda_2}\rangle) = \phi_1(\langle u_{\lambda_2}\rangle)\phi_1(\langle u_{\lambda_1}\rangle)$.
(2) ϕ_2 is a homomorphism, i.e $\phi_2(\langle u_{\lambda_1}\rangle\langle u_{\lambda_2}\rangle) = \phi_2(\langle u_{\lambda_1}\rangle)\phi_2(\langle u_{\lambda_2}\rangle)$.

Remark 3.5. *From the proposition above, we have $_{\alpha}\delta(\mathscr{C}(\Lambda)) \subseteq \mathscr{C}(\Lambda)$ and $\delta_\alpha(\mathscr{C}(\Lambda)) \subseteq \mathscr{C}(\Lambda)$ if $\langle u_\alpha \rangle \in \mathscr{C}(\Lambda)$. Moreover if $\langle u_\alpha\rangle$ is expressed as a combination of monomials of $\langle u_i \rangle, i \in I$, then $_{\alpha}\delta$ (resp. δ_α) can be expressed as the corresponding combination of monomials of $_{i}\delta$ (resp. δ_i), $i \in I$.*

4 Reflection functors and Lusztig's symmetries

Given a Cartan datum Δ as before, there is a valued graph (Γ, d) corresponding to it (where $\Gamma = (\Gamma_0, \Gamma_1)$, Γ_0 the set of vertices, Γ_1 the set of edges with $|\Gamma_0| = I$). We obtain (Γ, d, Ω) by prescribing an orientation Ω to (Γ, d), and always write Ω for simplicity. Then let $\mathscr{S} = (F_i, {}_iM_j)_{i.j \in \Gamma_0}$ be a reduced k-species of type Ω. Denote by rep-$\mathscr{S}$ the category of finite dimensional representations of $\mathscr{S}$ over k. We know that the category rep-$\mathscr{S}$ is equivalent to the module category of finite dimensional modules over a finite dimensional hereditary k-algebra Λ. This hereditary k-algebra Λ is given by the tensor algebra of $\mathscr{S}$ (See [DR]). Furthermore, any finite dimensional hereditary k-algebra can be obtained in this way.

Let p be a sink or a source of Ω. We define $\sigma_p\mathscr{S}$ to be the k-species obtained from $\mathscr{S}$ by replacing ${}_rM_s$ by its k-dual for $r = p$ or $s = p$, then $\sigma_p\mathscr{S}$ is a reduced k-species of type $\sigma_p\Omega$, where the orientation $\sigma_p\Omega$ is obtained by reversing the direction of arrows along all edges containing p.

We have the Bernstein-Gelfand-Ponomarev *reflection functors* $\sigma_p^{\pm}$: rep-$\mathscr{S}$ $\to$ rep-$\sigma_p\mathscr{S}$, see [BGP], [DR].

If i is a vertex of Γ, let rep-$\mathscr{S}\langle i \rangle$ be the subcategory of rep-$\mathscr{S}$ consisting of all representations which do not have V_i as a direct summand, where V_i is the simple representation corresponding to i. If i is a sink, then σ_i^+ : rep-$\mathscr{S}\langle i\rangle \to$

rep-$\sigma_i\mathscr{S}\langle i\rangle$ is an equivalence and it is exact and induces isomorphism on both Hom and Ext. The assertion for σ_i^- is the same if i is a source.

Let Λ just be the tensor algebra of a k-species $\mathscr{S}$. We can identify mod-Λ with rep-$\mathscr{S}$, therefore, $\mathscr{H}(\Lambda)$ can be viewed as being defined for rep-$\mathscr{S}$. We also use $\sigma_i\Lambda$ to denote the tensor algebra of $\sigma_i\mathscr{S}$ and $\sigma_i\Delta$ to denote the Cartan datum corresponding to the algebra $\sigma_i\Lambda$ (note that in fact $\sigma_i\Delta$ and Δ denote the same Cartan datum). We define $\mathscr{H}(\Lambda)\langle i\rangle$ to be the $\mathbb{Q}(v)$-subspace of $\mathscr{H}(\Lambda)$ generated by $\langle u_\alpha\rangle$ with $V_\alpha\in$ rep-$\mathscr{S}\langle i\rangle$. If i is a sink or a source, since rep-$\mathscr{S}\langle i\rangle$ is closed under extensions, $\mathscr{H}(\Lambda)\langle i\rangle$ is a subalgebra of $\mathscr{H}(\Lambda)$. The following result is due to Ringel [R5]:

Proposition 4.1. *Let i be a sink. The functor σ_i^+ yields a $\mathbb{Q}(v)$-algebra isomorphism $T_i:\mathscr{H}(\Lambda)\langle i\rangle\to\mathscr{H}(\sigma_i\Lambda)\langle i\rangle$ with $T_i(\langle u_\alpha\rangle)=\langle u_{\sigma_i^+\alpha}\rangle$ for any $V_\alpha\in$ rep-$\mathscr{S}\langle i\rangle$.*

The isomorphism T_i can be extended to the whole reduced Drinfeld double $\mathcal{D}(\Lambda)$ as follows (See [XY]):

Let $\overline{K_i}=v^{-\varepsilon(i)}K_i$, $\langle u_\alpha\rangle^{(t)}=\langle u_\alpha\rangle^t/([t]!)_{\varepsilon(\alpha)}$ for $\alpha\in\mathcal{P}$ and $t\in\mathbb{N}$.

For $\lambda\in\mathcal{P}$, assume that $V_\lambda=V_{\lambda_0}\oplus tV_i$ and V_{λ_0} contains no direct summand isomorphic to V_i. Then $\mathrm{Hom}_\Lambda(V_{\lambda_0},V_i)=0$ and $\mathrm{Ext}_\Lambda(V_i,V_{\lambda_0})=0$ since i is a sink of $\mathscr{S}$. Thus $\langle u_\lambda^+\rangle=v^{\langle\lambda_0,ti\rangle}\langle u_i^+\rangle^{(t)}\langle u_{\lambda_o}^+\rangle$ in $\mathscr{H}^+(\Lambda)$. We define T_i : $\mathscr{H}^+(\Lambda)\to\mathcal{D}(\sigma_i\Lambda)$ as follows:

$$T_i(\langle u_\lambda^+\rangle)=\frac{v^{\langle\lambda_0,ti\rangle}}{[t]!_{\varepsilon(i)}}(\langle u_i^-\rangle\overline{K_i})^t\langle u^+_{\sigma_i^+\lambda_0}\rangle=v^{\langle\lambda,ti\rangle}K_{ti}\langle u_i^-\rangle^{(t)}\langle u^+_{\sigma_i^+\lambda_0}\rangle.$$

In particular, $T_i(\langle u_i^+\rangle)=\langle u_i^-\rangle\overline{K_i}$.

Symmetrically we define a morphism $T_i:\mathscr{H}^-(\Lambda)\to\mathcal{D}(\sigma_i\Lambda)$ as follows:

$$T_i(\langle u_\lambda^-\rangle)=\frac{v^{\langle\lambda_0,ti\rangle}}{[t]!_{\varepsilon(i)}}(\overline{K_{-i}}\langle u_i^+\rangle)^t\langle u^-_{\sigma_i^+\lambda_0}\rangle=v^{\langle\lambda,ti\rangle}K_{-ti}\langle u_i^+\rangle^{(t)}\langle u^-_{\sigma_i^+\lambda_0}\rangle$$

for all $\lambda\in\mathcal{P}$, where $V_\lambda=V_{\lambda_0}\oplus tV_i$ and V_{λ_0} contains no direct summand isomorphic to V_i.

Also, we extend T_i to the torus algebra by setting $T_i(K_\alpha)=K_{s_i(\alpha)}$ for $\alpha\in\mathbb{Z}[I]$. Set $T_i(K_\alpha\langle u_\lambda^\pm\rangle)=T_i(K_\alpha)T_i(\langle u_\lambda^\pm\rangle)$.

The following theorem can be found in [XY].

Theorem 4.2. *Let i be a sink. The operator T_i induces a $\mathbb{Q}(v)$-algebra isomorphism: $\mathcal{D}_{\mathscr{C}}(\Delta)\xrightarrow{\sim}\mathcal{D}_{\mathscr{C}}(\sigma_i\Delta)$.*

If i is a source of $\mathscr{S}$, we can define T_i' (via the reflection functors σ_i^-) in the Hall algebra and extend it to $\mathcal{D}(\Lambda)$ similarly, which also induces a $\mathbb{Q}(v)$-algebra isomorphism: $\mathcal{D}_{\mathscr{C}}(\Delta)\to\mathcal{D}_{\mathscr{C}}(\sigma_i\Delta)$.

Recall that we have the isomorphism $\mathcal{D}_{\mathscr{C}}(\Delta)\xrightarrow{\sim}U$ in Theorem 3.1. So we have the canonical isomorphism $\mathcal{D}_{\mathscr{C}}(\Delta)\xrightarrow{\sim}\mathcal{D}_{\mathscr{C}}(\sigma_i\Delta)$ by mapping $\langle u_i^\pm\rangle\mapsto\langle u_i^\pm\rangle$ and $K_i\mapsto K_i$ for a sink $i\in I$. Therefore we can identify $\mathcal{D}_{\mathscr{C}}(\sigma_i\Delta)$ with $\mathcal{D}_{\mathscr{C}}(\Delta)$ under this canonical isomorphism. Then T_i induces an automorphism $\mathcal{D}_{\mathscr{C}}(\Delta)\xrightarrow{\sim}\mathcal{D}_{\mathscr{C}}(\Delta)$. Similarly T_i' can be viewed as an automorphism $\mathcal{D}_{\mathscr{C}}(\Delta)\xrightarrow{\sim}\mathcal{D}_{\mathscr{C}}(\Delta)$ for a source $i\in I$. The following theorem asserts that T_i and T_i' coincides with Lusztig's symmetries (see Section 2).

Theorem 4.3. *(1) Let i be s sink. Then the isomorphism $T_i : \mathcal{D}_{\mathscr{C}}(\Delta) \xrightarrow{\sim} \mathcal{D}_{\mathscr{C}}(\Delta)$ coincides with $T''_{i,1}$. Namely, $T_i = T''_{i,1}$, if we identify $\mathcal{D}_{\mathscr{C}}(\Delta)$ with U by Theorem 3.1.*

(2) Let i be a source. Then the isomorphism $T'_i : \mathcal{D}_{\mathscr{C}}(\Delta) \xrightarrow{\sim} \mathcal{D}_{\mathscr{C}}(\Delta)$ coincides with $T'_{i,-1}$.

We keep the notations before. Recall that a sequence $i_1, \cdots, i_m$ is called a *sink sequence* for Ω, provided i_1 is a sink for Ω, and for $1 < t \le m$, the vertex i_t is a sink for the orientation $\sigma_{i_{t-1}} \cdots \sigma_{i_1}\Omega$. The definition of a *source sequence* is similar. The following proposition comes from [R5].

Proposition 4.4. *If we identify $\mathcal{D}_{\mathscr{C}}(\Delta)$ with U, then:*

(1)For any preinjective module V_α, there exists a source sequence $i_1, \cdots, i_m$ for Ω such that

$$\langle u_\alpha \rangle = \langle u_{\sigma^+_{i_1} \cdots \sigma^+_{i_{m-1}} i_m} \rangle = T_{i_1} \cdots T_{i_{m-1}} \langle u_{i_m} \rangle = T''_{i_1,1} \cdots T''_{i_{m-1},1}(E_{i_m}).$$

(2)For any preinjective module V_α, there exists a sink sequence $i_1, \cdots, i_m$ for Ω such that

$$\langle u_\alpha \rangle = \langle u_{\sigma^-_{i_1} \cdots \sigma^-_{i_{m-1}} i_m} \rangle = T'_{i_1} \cdots T'_{i_{m-1}} \langle u_{i_m} \rangle = T'_{i_1,-1} \cdots T'_{i_{m-1},-1}(E_{i_m}).$$

5 The elements in Hall algebras corresponding to exceptional modules

Let Λ be a finite dimensional hereditary k-algebra as in Section 3. Denote by mod-Λ the category of finite-dimensional Λ-modules.

Recall that a Λ-module V_α is called *exceptional* if $\mathrm{Ext}^1_\Lambda(V_\alpha, V_\alpha) = 0$. A pair of indecomposable exceptional modules (V_α, V_β) is called an *exceptional pair* if $\mathrm{Hom}_\Lambda(V_\beta, V_\alpha) = \mathrm{Ext}^1_\Lambda(V_\beta, V_\alpha) = 0$. A sequence of indecomposable Λ-modules $(V_{\alpha_1}, V_{\alpha_2}, ..., V_{\alpha_n})$ is called an *exceptional sequence* if any pair $(V_{\alpha_i}, V_{\alpha_j})$ with $i < j$ is exceptional. An exceptional sequence $(V_{\alpha_1}, V_{\alpha_2}, ..., V_{\alpha_n})$ is said to be *complete* if $n = |I|$.

By Crawley-Boevey [CB] and Ringel [R2] we know that there is a nice braid group action on the set of complete exceptional sequences by which we can obtain all exceptional modules from an exceptional sequence consisting of simple modules. We will briefly recall the theory.

The braid group action is based on the following results: For any exceptional sequence $(V_{\alpha_1}, V_{\alpha_2}, ..., V_{\alpha_s})$, let $\mathcal{C}(\alpha_1, \alpha_2, ..., \alpha_s)$ be the smallest full subcategory of mod-Λ which contains $V_{\alpha_1}, V_{\alpha_2}, ..., V_{\alpha_s}$ and is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms. $\mathcal{C}(\alpha_1, \alpha_2, ..., \alpha_s)$ is equivalent to mod-Λ' where Λ' is a finite dimensional hereditary algebra with s isomorphism classes of simple modules. Furthermore, the canonical embedding of $\mathcal{C}(\alpha_1, \alpha_2, ..., \alpha_s)$ into mod-Λ is exact and induces isomorphisms on both Hom and Ext.

In particular, the results above holds for any exceptional pair (V_α, V_β). That is, $\mathcal{C}(\alpha, \beta)$ is equivalent to the module category of a generalized Kronecker algebra which has no regular exceptional modules. Hence (V_α, V_β) have to be slice modules in the preprojective component or preinjective component of $\mathcal{C}(\alpha, \beta)$ or

the orthogonal pair, i.e. V_α is the simple injective and V_β is the simple projective. Thus, for an exceptional pair (V_α, V_β), there are unique modules $L(\alpha,\beta)$ and $R(\alpha,\beta)$ such that $(L(\alpha,\beta), V_\alpha)$ and $(V_\beta, R(\alpha,\beta))$ are exceptional pairs in $\mathcal{C}(\alpha,\beta)$.

Let $\mathcal{V} = (V_{\alpha_1}, V_{\alpha_2}, ..., V_{\alpha_s})$ be an exceptional sequence in mod-Λ. For $1 \leq i \leq s$, Define $\sigma_i \mathcal{V} = (V_{\beta_1}, V_{\beta_2}, ..., V_{\beta_s})$, where

$$V_{\beta_j} = \begin{cases} V_{\alpha_{i+1}} & \text{if } j = i \\ R(\alpha_i, \alpha_{i+1}) & \text{if } j = i+1 \\ V_{\alpha_j} & \text{if } j \notin \{i, i+1\} \end{cases}$$

Also, define $\sigma_i^{-1}\mathcal{V} = (V_{\gamma_1}, V_{\gamma_2}, ..., V_{\gamma_s})$, where

$$V_{\gamma_j} = \begin{cases} L(\alpha_i, \alpha_{i+1}) & \text{if } j = i \\ V_{\alpha_i} & \text{if } j = i+1 \\ V_{\alpha_j} & \text{if } j \notin \{i, i+1\} \end{cases}$$

Denote by $\mathcal{B}_{s-1}$ the group generated by $\sigma_1, \sigma_2, ..., \sigma_{s-1}$. The above definitions give an action of $\mathcal{B}_{s-1}$ on the set of exceptional sequences of length s. Moreover, $\sigma_1, \sigma_2, ..., \sigma_{s-1}$ satisfy the braid relations:

$$\begin{cases} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } 1 \leq i \leq s-1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geq 2 \end{cases}$$

So $\mathcal{B}_{s-1}$ is the braid group of $s-1$ generators. In particular, in the case $s = |I|$, we have the braid group action on the set of complete exceptional sequences. This action is transitive according to Crawley-Boevey [CB] and Ringel [R2].

Since any indecomposable exceptional module can be enlarged to a complete exceptional sequence, we could obtain all indecomposable exceptional modules via the braid group action from any given complete exceptional sequence, in particular, the exceptional sequence consisting of all the simple modules. In [CX], an explicit inductive algorithm is given to express $\langle u_\lambda \rangle$ as the combinations of elements $\langle u_i \rangle$ if V_λ is an indecomposable exceptional module. We write down the formulas with some modifications, for the definition of ${}_\alpha\delta$ and δ_α in Section 3 is different from that in [CX].

Theorem 5.1. *For $1 \leq s \leq |I|$, let $\mathscr{B} = \langle \sigma_1, \sigma_2, ..., \sigma_{s-1} \rangle$ be the braid group on $s-1$ generators, $\mathcal{V} = (V_{\alpha_1}, V_{\alpha_2}, ..., V_{\alpha_s})$ any exceptional sequence of length s in mod-Λ. Denote by*

$$m(i, i+1) = \frac{\langle \alpha_i, \alpha_{i+1} \rangle}{\langle \alpha_{i+1}, \alpha_{i+1} \rangle} = 2\frac{(\alpha_i, \alpha_{i+1})}{(\alpha_{i+1}, \alpha_{i+1})}$$

and

$$n(i, i+1) = \frac{\langle \alpha_i, \alpha_{i+1} \rangle}{\langle \alpha_i, \alpha_i \rangle} = 2\frac{(\alpha_i, \alpha_{i+1})}{(\alpha_i, \alpha_i)}$$

and assume that $\sigma_i \mathcal{V} = (V_{\beta_1}, V_{\beta_2}, ..., V_{\beta_s})$, $\sigma_i^{-1}\mathcal{V} = (V_{\gamma_1}, V_{\gamma_2}, ..., V_{\gamma_s})$ for $1 \leq i \leq s-1$.

Then, in the Hall algebra $\mathscr{H}(\Lambda)$, we have

(1) *If* $m(i,i+1)\underline{\dim}V_{\alpha_{i+1}} > \underline{\dim}V_{\alpha_i}$, *then*

$$\langle u_{\beta_{i+1}}\rangle = \sum_{r=0}^{m(i,i+1)-1} (-1)^r v^{2dimV_{\alpha_i}} v^{\varepsilon(\alpha_i)} (v^{-\varepsilon(\alpha_{i+1})})^{(m(i,i+1)^2-m(i,i+1)r+r)}$$
$$\times\langle u_{\alpha_{i+1}}\rangle^{(r)}\delta_{\alpha_i}(\langle u_{\alpha_{i+1}}\rangle^{(m(i,i+1)-r)}).$$

(2) *If* $0 < m(i,i+1)\underline{\dim}V_{\alpha_{i+1}} < \underline{\dim}V_{\alpha_i}$, *then*

$$\langle u_{\beta_{i+1}}\rangle = \frac{v^{2m(i,i+1)\,dimV_{\alpha_{i+1}}}}{[m(i,i+1)]!_{\varepsilon(\alpha_{i+1})}}({}_{\alpha_{i+1}}\delta)^{m(i,i+1)}(\langle u_{\alpha_i}\rangle).$$

(3) *If* $m(i,i+1) \le 0$, *then*

$$\langle u_{\beta_{i+1}}\rangle = \sum_{r=0}^{-m(i,i+1)} (-1)^r (v^{-r\varepsilon(\alpha_{i+1})})\langle u_{\alpha_{i+1}}\rangle^{(r)}\langle u_{\alpha_i}\rangle\langle u_{\alpha_{i+1}}\rangle^{(-m(i,i+1)-r)}.$$

(1') *If* $n(i,i+1)\underline{\dim}V_{\alpha_i} > \underline{\dim}V_{\alpha_{i+1}}$, *then*

$$\langle u_{\gamma_i}\rangle = \sum_{r=0}^{n(i,i+1)-1} (-1)^r v^{2dimV_{\alpha_{i+1}}} v^{\varepsilon(\alpha_{i+1})} (v^{-\varepsilon(\alpha_i)})^{(n(i,i+1)^2-n(i,i+1)r+r)}$$
$$\times({}_{\alpha_{i+1}}\delta(\langle u_{\alpha_i}\rangle^{(n(i,i+1)-r)}))\langle u_{\alpha_i}\rangle^{(r)}.$$

(2')*If* $0 < n(i,i+1)\underline{\dim}V_{\alpha_i} < \underline{\dim}V_{\alpha_{i+1}}$, *then*

$$\langle u_{\gamma_i}\rangle = \frac{v^{2n(i,i+1)\,dimV_{\alpha_i}}}{[n(i,i+1)]!_{\varepsilon(\alpha_i)}}(\delta_{\alpha_i})^{n(i,i+1)}(\langle u_{\alpha_{i+1}}\rangle).$$

(3')*If* $n(i,i+1) \le 0$, *then*

$$\langle u_{\gamma_i}\rangle = \sum_{r=0}^{-n(i,i+1)} (-1)^r (v^{-r\varepsilon(\alpha_i)})\langle u_{\alpha_i}\rangle^{-n(i,i+1)-r}\langle u_{\alpha_{i+1}}\rangle\langle u_{\alpha_i}\rangle^{(r)}.$$

By this theorem we can see that for any indecomposable exceptional module V_λ, $\langle u_\lambda\rangle$ lies in the generic composition algebra $\mathscr{C}(\Delta)$.

6 Main results

We keep the notations as before. In this section we will identify the quantum group U (resp. the positive part U^+, the $\mathbb{Z}$-form $U^+_{\mathbb{Z}}$) with the reduced Drinfeld double $\mathcal{D}_{\mathscr{C}}(\Delta)$ (resp. the generic composition algebra $\mathscr{C}(\Delta)$, the integral generic composition algebra $\mathscr{C}_{\mathbb{Z}}(\Delta)$).

In Section 5 we have seen that for any indecomposable exceptional Λ-module V_λ, $\langle u_\lambda\rangle$ lies in the quantum group. Our first result is a stronger assertion which says that $\langle u_\lambda\rangle$ lies in the integral form of the quantum group and here V_λ can be any (not only indecomposable) exceptional module.

Theorem 6.1. *Let Λ be a finite-dimensional hereditary k-algebra, V_λ an exceptional Λ-module, then $\langle u_\lambda\rangle$ lies in $U^+_{\mathbb{Z}}$.*

In Section 8 we will prove that $\langle u_\lambda\rangle \in L(\infty)$. So $\langle u_\lambda\rangle$ lies in $L_{\mathbb{Z}}(\infty)$. By abuse of language, we denote the image of $\langle u_\lambda\rangle$ in $L_{\mathbb{Z}}(\infty)/v^{-1}L_{\mathbb{Z}}(\infty)$ still by $\langle u_\lambda\rangle$. Our second result is

Theorem 6.2. *Let Λ be a finite-dimensional hereditary k-algebra. Then for any exceptional module V_λ,*

$$\langle u_\lambda\rangle \in B(\infty)\cup(-B(\infty)).$$

i.e. $\langle u_\lambda\rangle$ lies in the crystal basis up to a sign.

7 Proof of Theorem 6.1

Theorem 5.1 provides an inductive method to express $\langle u_\lambda\rangle$ as combinations of the Chevalley generators. If we can prove that in each step the coefficients of the formulas are in $\mathbb{Z}[v,v^{-1}]$, we are done immediately for proving Theorem 6.1. Unfortunately we could not achieve this since the derivations ${}_\alpha\delta$ and δ_α are not $U_{\mathbb{Z}}$-stable. Instead, we will use Lusztig's symmetries which is known to be $U_{\mathbb{Z}}$-stable.

First we introduce some notations. For any exceptional pair (V_α, V_β), the subcategory $\mathcal{C}(\alpha,\beta)$ (Recall section 5) is equivalent to mod-Λ' for some finite dimensional hereditary k'-algebra Λ' which has two simple modules. Then the corresponding Hall algebra $\mathscr{H}(\Lambda')$ is a subalgebra of $\mathscr{H}(\Lambda)$ and the composition algebra $\mathscr{C}(\Lambda')$ is a subalgebra of $\mathscr{C}(\Lambda)$ (note that the simple Λ'-modules viewed as Λ-modules are exceptional). Denote the Cartan datum of Λ' by Δ'. The generic composition algebra $\mathscr{C}(\Delta')$ is also a subalgebra of $\mathscr{C}(\Delta)$. Now denote the quantum group associated to Δ' by U'. Then we have an embedding $U'^+ \hookrightarrow U^+$. The discussion here means that an exceptional pair gives a sub-Hall algebra of $\mathscr{H}(\Lambda)$ which corresponds to a sub-quantum group (positive part) of U^+.

Thus everything in Section 3 and 4 works for Λ'. Suppose that k' is a finite field with $q' = (v')^2$ elements where $v' = v^a$ for some positive integers a. For $V_\gamma \in \mathcal{C}(\alpha,\beta)$ we use the notation $\langle u_\gamma\rangle' = (v')^{-\dim_{k'}V_\gamma+\varepsilon'(\gamma)}u_\gamma$ in $\mathscr{H}(\Lambda')$ and the derivations $\delta'_\gamma, {}_\gamma\delta'$ are well-defined. Denote the two simple Λ'-modules by V_{s_1}, V_{s_2}, the corresponding elements in $\mathscr{H}(\Lambda')$ by $\langle u_{s_1}\rangle'$ and $\langle u_{s_2}\rangle'$. We have the (relative) symmetries T_{s_1}, T_{s_2} and T'_{s_1}, T'_{s_2} as in Section 4.

Lemma 7.1. *For any $V_\gamma \in \mathcal{C}(\alpha,\beta)$, we have*

$$\begin{aligned} &(1) \quad \langle u_\gamma\rangle' = \langle u_\gamma\rangle; \\ &(2) \quad \delta_\gamma|_{\mathcal{C}(\alpha,\beta)} = \delta'_\gamma, \quad {}_\gamma\delta|_{\mathcal{C}(\alpha,\beta)} = {}_\gamma\delta', \end{aligned}$$

where $\delta_\gamma|_{\mathcal{C}(\alpha,\beta)}$ and ${}_\gamma\delta|_{\mathcal{C}(\alpha,\beta)}$ denote the restrictions of δ_γ and ${}_\gamma\delta$ to $\mathscr{H}(\Lambda')$.

Proof. (1) By definition

$$\langle u_\gamma\rangle = v^{-\dim_k V_\gamma+\varepsilon(\gamma)}u_\gamma.$$

The number

$$v^{-\dim_k V_\gamma+\varepsilon(\gamma)} = \frac{\sqrt{|\mathrm{End}_\Lambda(V_\gamma)|}}{\sqrt{|V_\gamma||\mathrm{Ext}_\Lambda(V_\gamma,V_\gamma)|}}$$

is unchanged whether we consider V_γ as a Λ-module or a Λ'-module since the embedding $\mathcal{C}(\alpha,\beta)\hookrightarrow$ mod-Λ induces isomorphisms on both Hom and Ext.

(2) We only prove the assertion for δ_γ. Recall the definition of δ_γ in 3.3.

$$\delta_\gamma(u_\lambda) = \sum_{\rho\in\mathcal{P}} v^{\langle\gamma,\rho\rangle} g^\lambda_{\gamma\rho} \frac{a_\rho}{a_\lambda} u_\rho.$$

Since $\mathcal{C}(\alpha,\beta)$ is closed under kernels of epimorphisms, we can see that if $V_\lambda \in \mathcal{C}(\alpha,\beta)$, those V_ρ with u_ρ occurring in the right hand side must be in $\mathcal{C}(\alpha,\beta)$.

Note that

$$v^{\langle\gamma,\rho\rangle} = \frac{\sqrt{|\mathrm{Hom}_\Lambda(V_\gamma, V_\rho)|}}{\sqrt{|\mathrm{Ext}_\Lambda(V_\gamma, V_\rho)|}}.$$

Hence the numbers $v^{\langle\gamma,\rho\rangle}$, $g^\lambda_{\gamma\rho}$, a_ρ and a_λ are unchanged whether we consider $V_\gamma, V_\rho, V_\lambda$ as Λ-modules or Λ'-modules. □

The following lemma is the key point of the proof.

Lemma 7.2. *For any indecomposable exceptional module V_λ, there exist a positive integer n, subcategories $\mathscr{C}_i$ $(1 \le i \le n)$ of mod-Λ and indecomposable exceptional Λ-modules V_{α_i} $(1 \le i \le n+1)$ where for each i, $\mathscr{C}_i \simeq$ mod-Λ_i (here Λ_i is a finite dimensional hereditary algebra having exactly two simple modules $S_{i,1}$ and $S_{i,2}$) such that*

$$\langle u_\lambda\rangle = \langle u_{\alpha_{n+1}}\rangle;$$

and for each $1 \le i \le n$,

$$\langle u_{\alpha_{i+1}}\rangle = T_{S_{i,i_1}} T_{S_{i,i_2}} \cdots T_{S_{i,i_{d_i}}}(\langle u_{\alpha_i}\rangle), \text{ or } \langle u_{\alpha_{i+1}}\rangle = T'_{S_{i,i_1}} T'_{S_{i,i_2}} \cdots T'_{S_{i,i_{d_i}}}(\langle u_{\alpha_i}\rangle)$$

$$V_{\alpha_i} = S_{i,1} \text{ or } S_{i,2} \quad \text{i.e. } \langle u_{\alpha_i}\rangle = \langle u_{S_{i,1}}\rangle \text{ or } \langle u_{S_{i,2}}\rangle$$

where $i_1, i_2, \cdots, i_{d_i} \in \{1,2\}$ for each $1 \le i \le n$.

Proof. If V_λ is simple, there is nothing to prove. So assume V_λ is not simple. Then by Ringel (See [R5] Section 8) there exists an exceptional sequence (V_λ, V_μ) or (V_μ, V_λ) such that V_λ is not simple in $\mathcal{C}(V_\lambda, V_\mu)$.

We know that $\mathcal{C}(V_\lambda, V_\mu)$ is isomorphic to a finite dimensional hereditary algebra Λ' with just two simple modules and V_λ viewed as a Λ'-module is preprojective or preinjective. Denote the simple Λ'-modules by V_{S_1} and V_{S_2}. By Proposition 4.4 we have

$$\langle u_\lambda\rangle = T_{S_{i_1}} T_{S_{i_2}} \cdots T_{S_{i_{m-1}}}(\langle u_{S_{i_m}}\rangle)$$

where $i_1, i_2, \cdots, i_m (i_j \in \{1,2\}, j = 1, \cdots, m)$ is a source sequence of the graph of Λ', or

$$\langle u_\lambda\rangle = T'_{S_{i_1}} T'_{S_{i_2}} \cdots T'_{S_{i_{m-1}}}(\langle u_{S_{i_m}}\rangle)$$

where $i_1, i_2, \cdots, i_m (i_j \in \{1,2\}, j = 1, \cdots, m)$ is a sink sequence of the graph of Λ'.

Note that in the formulas above we should use $\langle\ \rangle'$, but Lemma 7.1(1) told us $\langle\ \rangle' = \langle\ \rangle$.

The lemma follows by induction. □

Now we can prove Theorem 6.1.

First let us see that theorem 6.1 holds for any indecomposable exceptional module V_λ. By the above lemma, in each step of the induction we are in some sub-quantum group, say U'. The symmetries are $U'_{\mathbb{Z}}$-stable and the coefficient ring is $\mathbb{Z}[v', v'^{-1}]$ where $v' = v^a$ for some positive integer a. Obviously $\mathbb{Z}[v', v'^{-1}] \subset \mathbb{Z}[v, v^{-1}]$. Hence $\langle u_\lambda \rangle$ lies in the integral composition algebra. The only thing we may worry about is whether the calculation is generic. But Theorem 5.1 has ensured it. Thus $\langle u_\lambda \rangle$ is in $U_{\mathbb{Z}}^+$.

Next we consider the case $V_\lambda \simeq sV_\rho$ where V_ρ is indecomposable exceptional (we also write sV_ρ as $V_{s\rho}$). It is well known that (see [R5], for example) we have

$$\langle u_{s\rho} \rangle = \langle u_\rho \rangle^{(s)},$$

Note that Lusztig's symmetries are endomorphisms hence the right hand side is also in $U_{\mathbb{Z}}^+$ by Lemma 7.2.

Now we consider the general case. By induction we can reduce to the case that $V_\lambda \simeq V_{s\mu} \oplus V_{t\nu}$ where V_μ and V_ν are non-isomorphic indecomposable exceptional modules.

We need two lemmas. One gives the calculation of the filtration number $g_{\alpha\beta}^{\gamma}$, that is due to Riedtmann [Rie] and Peng [P].

Lemma 7.3. *For any Λ-modules V_α, V_β and V_γ*

$$g_{\alpha\beta}^{\gamma} = \frac{a_\gamma |Ext_\Lambda(V_\alpha, V_\beta)_{V_\gamma}|}{a_\alpha a_\beta |Hom_\Lambda(V_\alpha, V_\beta)|}$$

where $Ext_\Lambda(V_\alpha, V_\beta)_{V_\gamma}$ is the set of exact sequence in $Ext_\Lambda(V_\alpha, V_\beta)$ with middle term V_γ.

The other one tells us how to compute the automorphism groups of decomposable modules (See [CX] or [Z]).

Lemma 7.4. (1) *Let V_λ be an indecomposable Λ-module with $dim_k End_\Lambda(V_\lambda) = s$ and $dim_k rad End_\Lambda(V_\lambda) = t$, then*

$$a_\lambda = (v^{2(s-t)} - 1)v^{2t}$$

(2) *Let $V_\lambda \cong s_1 V_{\lambda_1} \oplus \cdots \oplus s_t V_{\lambda_t}$ such that $V_{\lambda_i} \ncong V_{\lambda_j}$ for any $i \neq j$, then*

$$a_\lambda = v^{2s} a_{s_1\lambda_1} \cdots a_{s_t\lambda_t},$$

where $s = \sum_{i \neq j} s_i s_j dim_k Hom_\Lambda(V_{\lambda_i}, V_{\lambda_j})$.

(3) *Let $V_\lambda = sV_\rho$ with $End_\Lambda(V_\rho) = F$ and F is an extension field of k, then*

$$a_\lambda = \prod_{0 \leq t \leq s-1} (d^s - d^t),$$

where $d = |F| = v^{2[F:k]}$.

Now we can see that

$$\begin{aligned}\langle u_{s\mu} \rangle \langle u_{t\nu} \rangle &= v^{-\langle t\nu, s\mu \rangle} g_{(s\mu)(t\nu)}^{s\mu \oplus t\nu} \langle u_{s\mu \oplus t\nu} \rangle \\ &= v^{-\langle t\nu, s\mu \rangle} \frac{a_{s\mu \oplus t\nu}}{a_{s\mu} a_{t\nu} |\mathrm{Hom}_\Lambda(V_{s\mu}, V_{t\nu})|} \langle u_{s\mu \oplus t\nu} \rangle \\ &= v^{-\langle t\nu, s\mu \rangle} |\mathrm{Hom}_\Lambda(V_{t\nu}, V_{s\mu})| \langle u_{s\mu \oplus t\nu} \rangle\end{aligned}$$

Hence we have

$$\langle u_\lambda\rangle = \langle u_{s\mu\oplus t\nu}\rangle = v^{\langle t\nu, s\mu\rangle - 2st\dim_k \mathrm{Hom}_\Lambda(V_\nu, V_\mu)}\langle u_{s\mu}\rangle\langle u_{t\nu}\rangle,$$

which lies in $U_{\mathbb{Z}}^+$ and we are done.

8 Proof of Theorem 6.2

8.1 The pairing $(-,-)_R$. Define a pairing $(-,-)_R$: $\mathscr{H}(\Lambda)\times\mathscr{H}(\Lambda)\to\mathbb{Q}(v)$ by

$$(\langle u_\beta\rangle, \langle u_{\beta'}\rangle)_R = v^{(\beta,\beta)}a_\beta^{-1}\delta_{\beta\beta'},$$

for all $\beta, \beta' \in \mathcal{P}$. (This pairing was first proposed by Ringel, see [R4])

Now we have two non-degenerate symmetric bilinear forms on U^+, namely $(-,-)_R$ and $(-,-)_K$ (recall Section 2). We will deduce some properties of $(-,-)_R$ and compare it with $(-,-)_K$. Note that in Lemma 3.3, we have proved that the derivations r_i' and f_i' coincide.

Denote $v^{\varepsilon(i)}$ by v_i. We have the following lemma.

Lemma 8.1. *For any $i\in I$ and $\lambda_1,\lambda_2\in\mathcal{P}$, we have*

$$(\langle u_{\lambda_1}\rangle, \langle u_i\rangle\langle u_{\lambda_2}\rangle)_R = (1-v_i^{-2})^{-1}(r_i'(\langle u_{\lambda_1}\rangle), \langle u_{\lambda_2}\rangle)_R,$$

$$(\langle u_{\lambda_1}\rangle, \langle u_{\lambda_2}\rangle\langle u_i\rangle)_R = (1-v_i^{-2})^{-1}(\langle u_{\lambda_1}\rangle, r_i(\langle u_{\lambda_2}\rangle))_R.$$

Proof. We only prove the first one. The proof of the second one is similar. By definition, we know that

$$\langle u_i\rangle\langle u_{\lambda_2}\rangle = v^{-\langle\lambda_2, i\rangle}\sum_{\lambda\in\mathcal{P}} g_{i\lambda_2}^{\lambda}\langle u_\lambda\rangle.$$

So we have

$$(\langle u_{\lambda_1}\rangle, \langle u_i\rangle\langle u_{\lambda_2}\rangle)_R = v^{(\lambda_1,\lambda_1)-\langle\lambda_2,i\rangle}\frac{g_{i\lambda_2}^{\lambda_1}}{a_{\lambda_1}}.$$

On the other hand, we have

$$r_i'(\langle u_{\lambda_1}\rangle) = \sum_{\beta\in\mathcal{P}} v^{\langle i,\beta\rangle+(i,\beta)}g_{i\beta}^{\lambda_1}\frac{a_\beta a_i}{a_{\lambda_1}}\langle u_\beta\rangle,$$

Thus we have

$$(r_i'(\langle u_{\lambda_1}\rangle), \langle u_{\lambda_2}\rangle)_R = v^{\langle i,\lambda_2\rangle+(i,\lambda_2)+(\lambda_2,\lambda_2)}\frac{g_{i\lambda_2}^{\lambda_1}a_i}{a_{\lambda_1}}.$$

Note that

$$(\lambda_1,\lambda_1) = (\lambda_2+i, \lambda_2+i) = (\lambda_2,\lambda_2)+2(\lambda_2,i)+(i,i)$$

and

$$a_i = (v_i^2-1) = v^{(i,i)}(1-v_i^{-2})^{-1},$$

we are done. ∎

By Lemma 8.1 we can compute the pairing $(-,-)_R$ inductively, similar to Proposition 2.2.

Lemma 8.2. *For any $x, y \in U^+$, we have*

$$
\begin{aligned}
(1,1)_R &= 1,\\
(E_i x, y)_R &= (1 - v_i^{-2})^{-1}(x, f_i'(y))_R.
\end{aligned}
$$

Proof. By definition and Lemma 8.1 we have

$$
\begin{aligned}
(E_i x, y)_R &= (\langle u_i \rangle x, y)_R = (1 - v_i^{-2})^{-1}(x, r_i'(y))_R\\
&= (1 - v_i^{-2})^{-1}(x, f_i'(y))_R.
\end{aligned}
$$

□

Lemma 8.3. *For any $x, y \in U^+$, we have $(x, y)_K \in A$ if and only if $(x, y)_R \in A$. In particular,*

$$
(L(\infty), L(\infty))_R \subset A.
$$

Proof. First we have

$$
(1,1)_R = (1,1)_K = 1.
$$

For any $x, y \in U^+$, by Lemma 8.2, we know

$$
(E_i x, y)_R = (1 - v_i^{-2})^{-1}(x, f_i'(y))_R.
$$

Since $(1 - v_i^{-2})^{-1}$ and $1 - v_i^{-2}$ are both in A, we have $(E_i x, y)_R \in A$ if and only if $(x, f_i'(y))_R \in A$. On the other hand. by Proposition 2.2,

$$
(E_i x, y)_K = (x, f_i'(y))_K.
$$

Now the assertion of this lemma follows immediately by induction. □

Hence the form $(-,-)_R$ induces a $\mathbb{Q}$-bilinear form $(-,-)_{R,0}$ on $L(\infty)/v^{-1}L(\infty)$:

$$
(x + v^{-1}L(\infty), y + v^{-1}L(\infty))_{R,0} = (x, y)_R + v^{-1}A
$$

for any $x, y \in L(\infty)$. The next proposition says that the two pairings $(x, y)_{R,0}$ and $(x, y)_{K,0}$ coincide.

Proposition 8.4. *For any $x, y \in L(\infty)$, if we denote their images in $L(\infty)/v^{-1}L(\infty)$ still by x, y, then we have*

$$
(x, y)_{R,0} = (x, y)_{K,0}.
$$

Proof. Just compare Proposition 2.2 and Lemma 8.3, and note that

$$
\frac{1}{1 - v_i^{-2}} = 1 + \frac{v_i^{-2}}{1 - v_i^{-2}} \in \; 1 + v^{-1}A.
$$

□

Thus we can use the pairing $(-,-)_R$ instead of $(-,-)_K$ to characterize the crystal bases.

Lemma 8.5. *We have the following characterizations of $L(\infty)$ and $B(\infty)$:*

$$L(\infty) = \{x \in U^+ | (x,x)_R \in A\},$$
$$B(\infty) \cup (-B(\infty)) = \{x \in L_{\mathbb{Z}}(\infty)/v^{-1}L_{\mathbb{Z}}(\infty) | (x,x)_{R,0} = 1\}.$$

Proof. Recall Proposition 2.3(b) and 2.4(c). The assertion of this corollary is an easy consequence of Lemma 8.3 and Proposition 8.4. $\square$

8.2 The proof. We need to calculate the pairing $(\langle u_\lambda\rangle, \langle u_\lambda\rangle)_R$ for each exceptional module V_λ.

First we assume that V_λ is indecomposable. In this case the calculation is easy since the endomorphism ring of V_λ is a division ring. namely

$$\begin{aligned}(\langle u_\lambda\rangle, \langle u_\lambda\rangle)_R &= \frac{v^{(\lambda,\lambda)}}{a_\lambda} = \frac{|\mathrm{End}_\Lambda(V_\lambda)|}{|\mathrm{Aut}_\Lambda(V_\lambda)|} \\ &= \frac{v^{2\varepsilon(\lambda)}}{v^{2\varepsilon(\lambda)}-1} = \frac{1}{1-v^{-2\varepsilon(\lambda)}} \\ &= 1 + \frac{v^{-2\varepsilon(\lambda)}}{1-v^{-2\varepsilon(\lambda)}} \ \in 1+v^{-1}A.\end{aligned}$$

Secondly, for n-copies of an indecomposable V_λ we get (using Lemma 7.4(3)):

$$\begin{aligned}(\langle u_\lambda\rangle^{(n)}, \langle u_\lambda\rangle^{(n)})_R = (\langle u_{n\lambda}\rangle, \langle u_{n\lambda}\rangle) &= \frac{|\mathrm{End}_\Lambda(V_{n\lambda})|}{a_{n\lambda}} \\ &= \frac{v^{2n^2\varepsilon(\lambda)}}{\prod_{0\le t\le n-1}(v^{2n\varepsilon(\lambda)} - v^{2t\varepsilon(\lambda)})} \\ &= \prod_{0\le t\le n-1} \frac{1}{1-v^{-2(n-t)\varepsilon(\lambda)}} \ \in 1+v^{-1}A.\end{aligned}$$

Now we can deal with any exceptional module V_λ. Assume $V_\lambda \cong s_1V_{\lambda_1} \oplus \cdots \oplus s_tV_{\lambda_t}$ such that V_{λ_i} is indecomposable for any i and $V_{\lambda_i} \not\cong V_{\lambda_j}$ for any $i \neq j$. Thus

$$|\mathrm{End}_\Lambda(V_\lambda)| = v^{2\sum_{i=1}^t s_i^2\varepsilon(\lambda_i)+2s}$$

where $s = \sum_{i\neq j} s_is_j \dim_k \mathrm{Hom}_\Lambda(V_{\lambda_i}, V_{\lambda_j})$.

By lemma 7.4 (2) we have $a_\lambda = v^{2s}a_{s_1\lambda_1}\cdots a_{s_t\lambda_t}$, hence

$$\begin{aligned}(\langle u_\lambda\rangle, \langle u_\lambda\rangle)_R &= \frac{v^{2\sum_{i=1}^t s_i^2\varepsilon(\lambda_i)+2s}}{v^{2s}a_{s_1\lambda_1}\cdots a_{s_t\lambda_t}} = \frac{v^{2\sum_{i=1}^t s_i^2\varepsilon(\lambda_i)}}{a_{s_1\lambda_1}\cdots a_{s_t\lambda_t}} \\ &= \frac{v^{2\sum_{i=1}^t s_i^2\varepsilon(\lambda_i)}}{\prod_{i=1}^t\prod_{0\le t_i\le s_i-1}(v^{2s_i\varepsilon(\lambda_i)} - v^{2t_i\varepsilon(\lambda_i)})} \\ &= \prod_{i=1}^t \prod_{0\le t_i\le s_i-1} \left(\frac{1}{1-v^{-2(s_i-t_i)\varepsilon(\lambda_i)}}\right) \ \in 1+v^{-1}A.\end{aligned}$$

Thus we have proved that for any exceptional module V_λ,

$$(\langle u_\lambda\rangle, \langle u_\lambda\rangle)_R \in A \ \text{ and } \ (\langle u_\lambda\rangle, \langle u_\lambda\rangle)_{R,0} = 1.$$

Hence by Theorem 6.1 and Lemma 8.5 we reach our goal.

Remark 8.6. *(1). In the rank 2 case (i.e. $|I| = 2$ in the Cartan datum), the sign can be removed (See [S] and [L2]). However, it seems that their methods do not work in general.*

(2). The formulas given in Theorem 5.1 can be viewed as an inductive algorithm to obtain certain crystal basis elements from the Chevalley generators.

9 Remove the sign

We have shown by algebraic methods that the elements corresponding to exceptional modules lie in the crystal bases up to a sign. Intuitively the sign should be removed since it does for rank 2 case. In this section we will remove the sign with geometric methods due to Lusztig. For convenience, we only consider the case of symmetric Cartan datum.

So let us consider a quiver without cycles, that is $Q = (I, H, s, t)$, where I is the vertex set, H is the arrow set and two maps s, t indicate the start points and terminal points of arrows respectively. let F_q denotes a finite field of $q = p^e$ elements, where p is a prime number. As in 3.1, we get a Hall algebra from the representation of $\Lambda = F_q Q$, hence a realization of U^+.

Now we give a short review of two constructions by Lusztig. The first is the geometrical construction of Hall algebras. For any finite dimensional I-graded F_q-vector space $W = \sum_{i \in I} W_i$, consider the moduli space of representation of Q:

$$E_W = \bigoplus_{\rho \in H} \mathrm{Hom}(W_{s(\rho)}, W_{t(\rho)})$$

The group $G_W = \prod_{i \in I} GL(W_i)$ acts on E_W naturally. Let $\mathbb{C}_G(E_W)$ be the space of G_W-invariant functions $E_W \to \mathbb{C}$. For $\underline{c} \in \mathbb{N}I$, we fix a I-graded F_q-vector space $W_{\underline{c}}$ with $\underline{\dim} W_{\underline{c}} = \underline{c}$, and denote $E_{\underline{c}} = E_{W_{\underline{c}}}, G_{\underline{c}} = G_{W_{\underline{c}}}$. Then the multiplication can be defined in the $\mathbb{C}$-space $K = \bigoplus_{\underline{a} \in \mathbb{N}I} \mathbb{C}_G(E_{\underline{a}})$, which makes K an associative $\mathbb{C}$-algebra. Corresponding to $\alpha \in \mathcal{P}$, let $\mathcal{O}_\alpha \subset E_{\underline{a}}$ be the $G_{\underline{a}}$-orbit of module $V_\alpha \in E_{\underline{a}}$. We take $1_\alpha \in \mathbb{C}_G(E_{\underline{a}})$ to be the characteristic function of $\mathcal{O}_\alpha$, and set $f_\alpha = v_q^{-\dim \mathcal{O}_\alpha} 1_\alpha$ where $v_q = \sqrt{q}$. Thus K is just the so-called Hall algebra if f_α is identified with $\langle u_\alpha \rangle$ for all $\alpha \in \mathcal{P}$.

The second is the construction of U^+ in terms of perverse sheaves. E_W can be defined over an algebraic closure of the finite field F_p of p elements. E_W has a natural F_{p^e}-structure with Frobenius map: $F^e : E_W \to E_W$. Thus the F_q-rational points $E_W^{F^e}$ of E_W provide an F_q-structure as above. Let $D(E_W) = D_c^b(E_W)$ be the bounded derived category of $\overline{\mathbb{Q}_l}$- constructible sheaves; here l is a fixed prime number distinct from p and $\overline{\mathbb{Q}_l}$ is an algebraic closure of the field of l-adic numbers. Let P_W be the set of isomorphism classes of simple perverse sheaves subject to some extra condition (see [L1]). Then we can associate P_W a full subcategory Q_W of $D(E_W)$ and let K_W be the Grothendieck group of Q_W. Hence $\coprod_W K_W$ gives a realization of U^+ with $\coprod_W P_W$ as its bases, that is the canonical bases.

For an exceptional module V_λ, $\mathcal{O}_\lambda$ is the corresponding orbit. We know from [L3] that the intersection cohomology complex

$$IC(\mathcal{O}_\lambda, \overline{\mathbb{Q}_l}) = j_{!*}(\overline{\mathbb{Q}_l})[\dim \mathcal{O}_\lambda]$$

belongs to the canonical bases P_W of K_W where j is the natural embedding $\mathcal{O}_\lambda \to E_W$.

In [L4], Lusztig considered the correspondence between functions on the moduli space and the elements of the canonical bases. In other words, he showed what kind of functions lies in the canonical bases when comparing the two constructions mentioned above. We will use the notations in [L4] from now on.

We only need to consider a special kind of canonical bases, i.e. $b = IC(\mathcal{O}_\lambda, \overline{\mathbb{Q}_l})$. b corresponds to $(b_e)_{e\geq 1}$, $b_e : E_W^{F^e} \to \overline{\mathbb{Q}_l}$. Sometimes $\overline{\mathbb{Q}_l}$ can be identified with $\mathbb{C}$. For $x \in E_W^{F^e}$, $b_e(x)$ is the alternative sum of the trace of the induced Frobenius map on the stalk at x of the i-th cohomology sheaf of $IC(\mathcal{O}_\lambda, \overline{\mathbb{Q}_l})$.

$$b_e(x) = \sum_{i\in\mathbb{Z}} (-1)^i Tr(F^e; H_x^i(j_{!*}(\overline{\mathbb{Q}_l})[\mathrm{dim}\mathcal{O}_\lambda])$$

Now that $\mathcal{O}_\lambda$ is an open dense subset of E_W, we can deduce as follows.

Firstly, the result below is due to Gabber:

$$Tr(F^e; H_x^i(j_{!*}(\overline{\mathbb{Q}_l})[\mathrm{dim}\mathcal{O}_\lambda]) \in q^{-\frac{\mathrm{dim}\mathcal{O}_\lambda}{2}}\mathbb{Z}[q^{-1}]$$

For $x \in E_W$, $F^e(x) = x \in \mathcal{O}_\lambda$, we have

$$Tr(F^e; H_x^i(j_{!*}(\overline{\mathbb{Q}_l})[\mathrm{dim}\mathcal{O}_\lambda]) = Tr(F^e; H_x^i(\mathcal{O}_\lambda, \overline{\mathbb{Q}_l}[\mathrm{dim}\mathcal{O}_\lambda]) = q^{-\frac{\mathrm{dim}\mathcal{O}_\lambda}{2}}\delta_{i,dim\mathcal{O}_\lambda}$$

While for $y \in E_W$, $F^e(y) = y \notin \mathcal{O}_\lambda$, any open neighborhood U_y of y,

$$U_y \cap \mathcal{O}_\lambda = U'_y \neq \varnothing \Longrightarrow H_y^i(j_{!*}(\overline{\mathbb{Q}_l})[\mathrm{dim}\mathcal{O}_\lambda]) \cong H_x^i(j_{!*}(\overline{\mathbb{Q}_l})[\mathrm{dim}\mathcal{O}_\lambda])$$

So b_e is actually a constant function

$$b_e = q^{-\frac{\mathrm{dim}\mathcal{O}_\lambda}{2}}(-1)^{\mathrm{dim}\mathcal{O}_\lambda} = (-q^{\frac{1}{2}})^{-\mathrm{dim}\mathcal{O}_\lambda}$$

We remark that $v = -q^{\frac{1}{2}}$ in the literature of [L4], so $b_e = v^{-\mathrm{dim}\mathcal{O}_\lambda}$. let $\mathcal{O}_y$ be the orbit of y, we obtain:

$$b_e|_{\mathcal{O}_\lambda} = v^{-\mathrm{dim}\mathcal{O}_\lambda}1_{\mathcal{O}_\lambda}$$

$$b_e|_{\mathcal{O}_y} = v^{-\mathrm{dim}E_W}1_{\mathcal{O}_y} = v^{-(\mathrm{dim}E_W - \mathrm{dim}\mathcal{O}_y)}v^{-\mathrm{dim}\mathcal{O}_y}1_{\mathcal{O}_y}$$

This means the image of b_e in $L_\mathbb{Z}(\infty)/v^{-1}L_\mathbb{Z}(\infty)$ is equal to that of $\langle u_\lambda\rangle$ in $L_\mathbb{Z}(\infty)/v^{-1}L_\mathbb{Z}(\infty)$. Hence we obtain the finally result, which is a stronger version of Theorem 6.2.

Theorem 9.1. *Let Λ be a finite-dimensional hereditary k-algebra. Then for any exceptional module V_λ, we have*

$$\langle u_\lambda\rangle \in B(\infty).$$

i.e. the image of $\langle u_\lambda\rangle$ in $L_\mathbb{Z}(\infty)/v^{-1}L_\mathbb{Z}(\infty)$ lies in the crystal basis.

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